Chapter 2: Probability and Statistics

Jonathan Roth

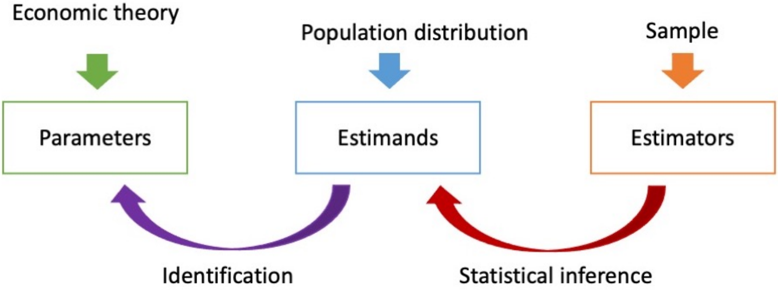
Mathematical Econometrics I
Brown University

Course Logistics

- Problem set 1 is posted. It is due on Friday September 19 at 4PM as a GradeScope submission
- TA sessions and OHs start this week. See the Canvas page for times/tentative locations
- Any logistical questions?

“Big Picture” Recap

Recall our division of labor:



- **Statistics**: how does the sample data we observe relate to observable features of the population we're interested in?
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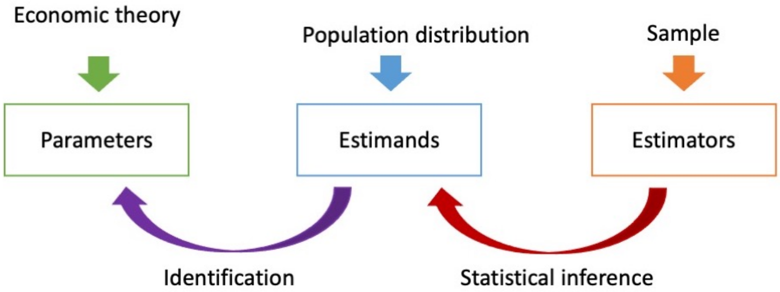

- **Statistics**: how does the sample data we observe relate to observable features of the population we're interested in?
- **Identification**: how do observable features of population relate to target parameters of interest?
- For both these tasks, we need a mathematical language for talking about how data is generated. Enter **probability and statistics**


I love
Statis- sticks

Outline

1. Random Variables and Probability Distributions
2. Means and Variances
3. Identification in Experiments
4. Random Sampling and Sample Means
5. Hypothesis Testing and Inference

Random Variables

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- What are some examples of a random process?
  - Flip a coin – is it heads or tails?
  - Survey a random household in US – what is their income?
- The realization of a random process is called a **random variable**.

Some Terminology

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- What are the possible outcomes (sample space)?
- What is the probability of seeing at least one head (an event)?

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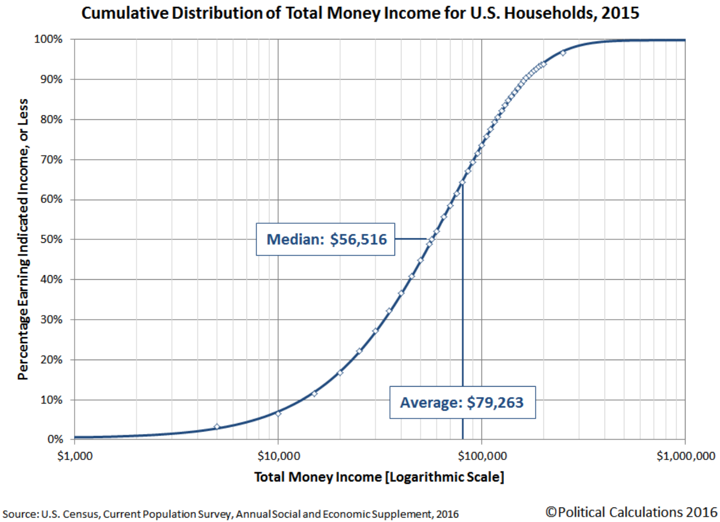
    - Note: we'll typically use lower-case letters like "x" to denote realizations (i.e. non-random numbers) of random variables like "X" ...



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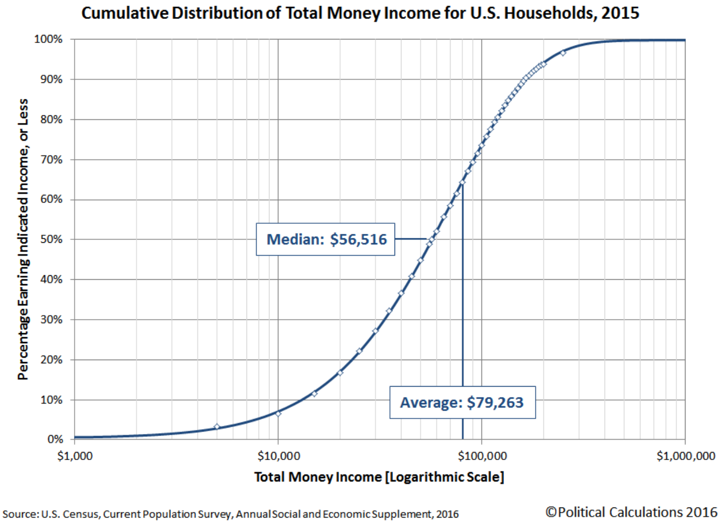

- In 2016, around half of US households earned \$56K or less
- Formally: $F(56,516) = 0.5$

We All Need Some Support...

- The **support** of a random variable X, denoted $\mathbb{X}$, is the set of values that X can take
  - If X is months in the year you were employed, $\mathbb{X} = \{0, 1, ..., 12\}$
  - If X is your income, then $\mathbb{X} = \mathbb{R}_{\geq 0}$ (approximately)

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- If the support of X is finite (e.g. $\{0, 1\}$), we say X is **discrete**
- If the support of X is a continuum (e.g. $\mathbb{R}$ or $[0, 1]$), we say X is **continuously distributed** (technically, if the CDF is differentiable)

Density and Mass Functions

- If X is discrete, we define the probability mass function (PMF) as the probability that X takes on each value in the support:

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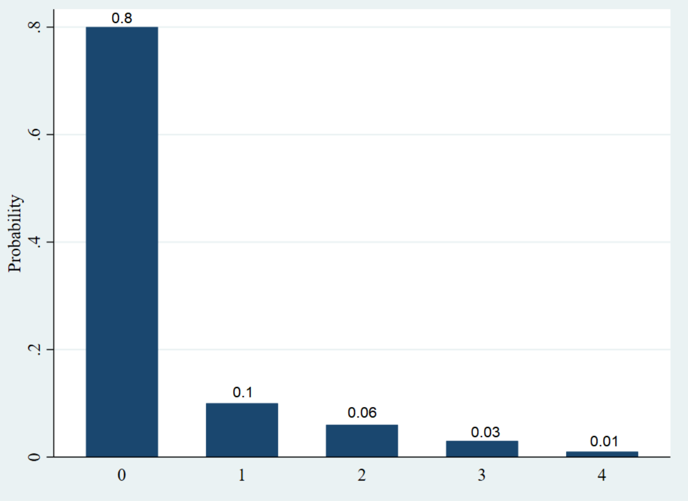
- Notational note: both $p(x)$ and $f(x)$ are used for PDFs/PMFs

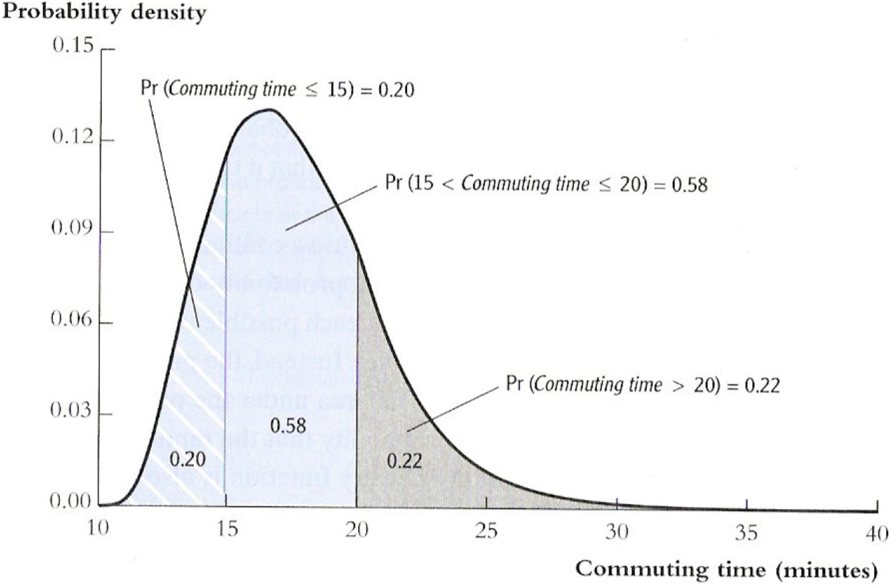
Example of a discrete random variable: number of wifi connection failures

- Here's the PMF; what is the CDF?



Example of a continuous random variable: commuting time

- Here's the PDF; what is the CDF?



Properties of PDFs/CDFs

- Key properties of CDFs $F(x) = Pr(X \leq x)$:
  - Non-decreasing: $F(x) \geq F(x')$ if $x > x'$
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- Corresponding properties of PDFs $f(x) = \frac{\partial}{\partial x} F(x)$:
  - Non-negative: $f(x) \geq 0$ for all $x \in \mathbb{X}$
  - Satisfies $\int_{x \in \mathbb{X}} f(x) dx = 1$
  - For PMFs: $\sum_{x \in \mathbb{X}} p(x) = 1$

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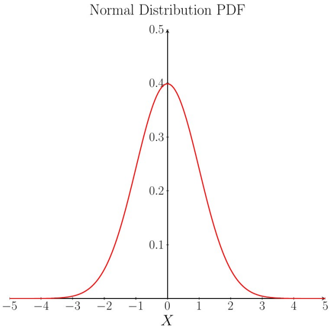
- Note this is the *only* distribution of any binary X

Normal and Uniform Distributions

An important continuous distribution: Normal $X \in \mathbb{R}$

- Example: the log of annual income (approximately)
- PDF: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$ for $x \in \mathbb{R}$ and $\sigma > 0$
- Here μ is the mean of X and σ^2 is its variance (also defined soon)
- Written $X \sim \mathrm{N}(\mu, \sigma^2)$
- Useful property: if X is normally distributed then so is $aX + b$ for any non-random a and b



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- Written $X \sim \mathrm{U}(a, b)$; also closed under linear transformations

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- Linked to the joint distribution by, e.g., $p(x) = \sum_{y \in \mathbb{Y}} p(x,y)$

Conditional Distributions

Combining joint and marginal distributions gives us the **conditional distribution** of one random variable given another

- Intuitively, the conditional distribution $Y|X = x$ is the distribution of Y among the sub-population with $X = x$
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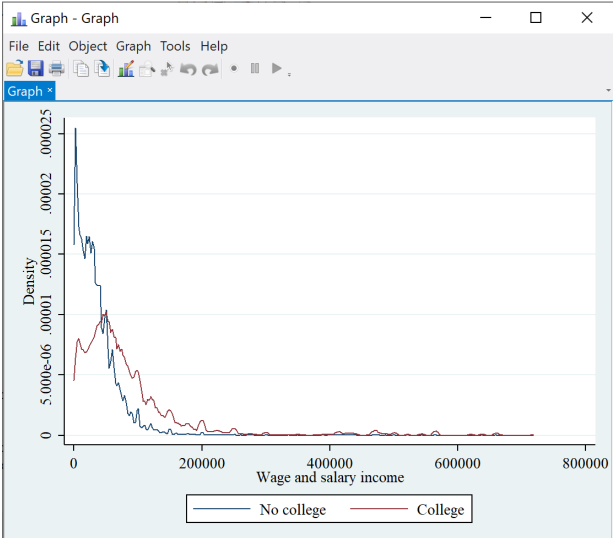
Leads immediately to *Bayes' rule*:

$$p(y \mid x) = p(x \mid y)\frac{p(y)}{p(x)}$$

Random Variables and Probability Distributions XI

Conditional PDFs of annual income given college completion:

```
twoway (kdensity incwage if educ<10) (kdensity incwage if educ>=10), ///
    xtitle("Wage and salary income") ytitle("Density") ///
    legend(label(1 "No college") label(2 "College"))
```



Independence

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- Intuitively, independence says that knowing the value of X tells us nothing about the value of Y
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- **Conditional independence** is defined similarly. $Y \perp\!\!\!\perp X \mid W$ if

$$p(y \mid x, w) = p(y|w), \forall(y, x, w)$$

- Intuitively, X tells us nothing about Y once we know W.

Multivariate Normals

An important multivariate distribution: joint normal $\mathbf{X} \in \mathbb{R}^K$

- Parameterized by a (mean) vector $\boldsymbol{\mu} \in \mathbb{R}^K$ and a positive-definite (variance-covariance) matrix $\boldsymbol{\Sigma} \in \mathbb{R}^K \times \mathbb{R}^K$
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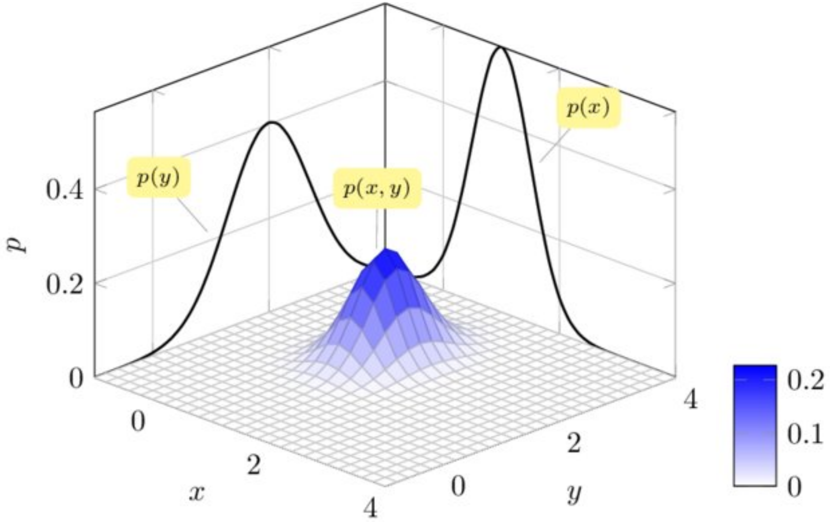
Many useful facts; here's a few. If $(X, Y)'$ is joint-normally distributed:

- The marginal distributions of X and Y are normal
- The conditional distributions of $X \mid Y$ and $Y \mid X$ are normal
- Any fixed linear combination $aX + bY + c$ is normally distributed

Bivariate Normal PDF



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  - For continuous random variables, $E[X] = \int_{x \in \mathbb{X}} f(x)x dx$

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- **Important fact:** The expectation operator is *linear*:
  $E[a + bX] = a + bE[X]$ for constants (a, b)
  - Easily proved from the above definitions (make sure you can!)

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- Plugging this in, we have

$$E[X] = \frac{1}{6}(1 + ... + 6) = 3.5$$

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This implies that $Std(a + bX) = b \cdot Std(X)$.

- Intuitively, if I measure income in cents, the standard deviation should be 100 times if I measure it in dollars

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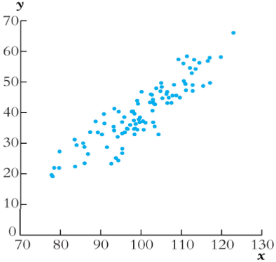
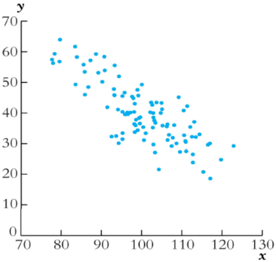
- But <u>not</u> vice-versa! Independence is a stronger notion of association

Examples of Correlations



(a) Correlation = +0.9



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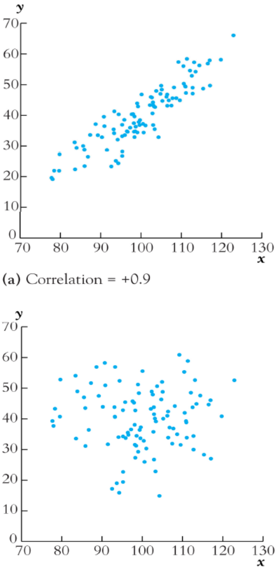

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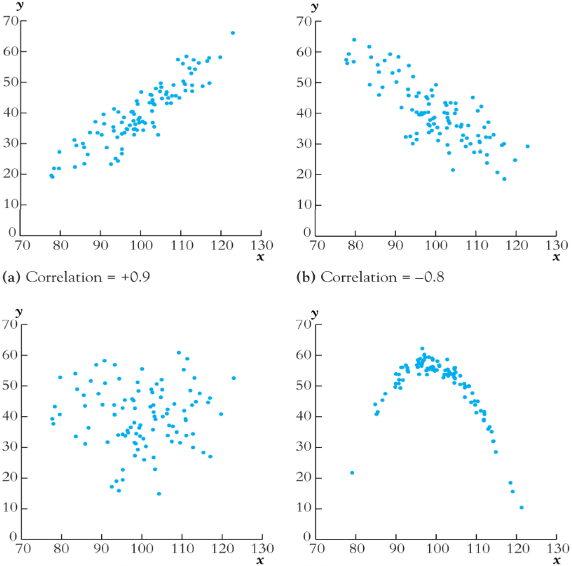

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- Covariances are linear: $Cov(aX + c, bY + d) = abCov(X, Y)$
  and $Cov(X + Z, Y) = Cov(X, Y) + Cov(Z, Y)$

Conditional Expectations

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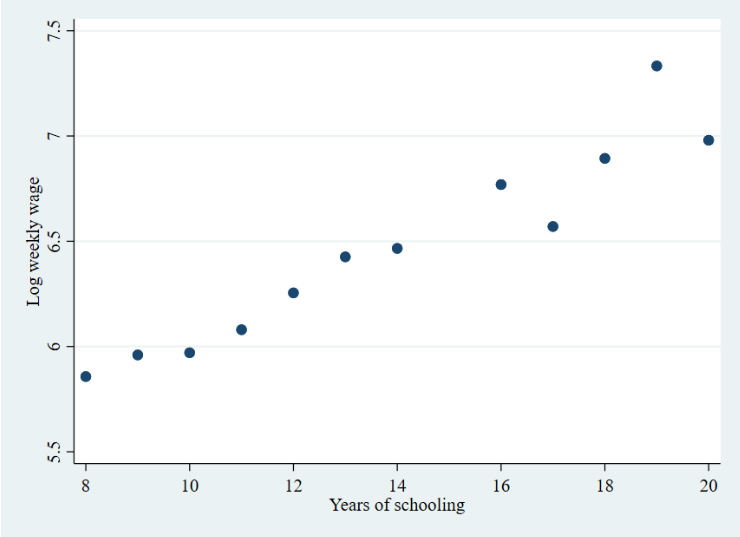
- Say Y is *mean independent* of X when $E[Y \mid X = x] = E[Y]$ for all x
- Conditioning on X makes functions of it constant: e.g.
  $E[f(X) + g(X)Y|X = x] = f(x) + g(x)E[Y|X = x]$ for any $f(\cdot)$, $g(\cdot)$

Conditional Expectation Example

CEF of (log) annual income given years of schooling



The Big LIE

A very important result for us: the **Law of Iterated Expectations** (LIE)

Let's start with an example. Suppose I want to calculate the average height of people in the United States. The LIE says I can:

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Note that the expectation on the LHS uses $p(y)$, while the outer expectation on the RHS uses $p(x)$ and the inner expectation uses $p(y \mid x)$

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The LIE shows us that mean independence implies uncorrelatedness

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Make sure you understand how we got each step!

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- Also, of course, independent $\implies$ mean independent (but not $\impliedby$)

Quick Aside on Vector/Matrix Notation

Often it will be useful to work with random vectors $\mathbf{X} = [X_1, \ldots, X_N]'$

- These will always be "columns" in this course, and denoted in bold
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Expectations are elementwise: e.g. $E\left[\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\right] = \begin{bmatrix} E[X_{11}] & E[X_{12}] \\ E[X_{21}] & E[X_{22}] \end{bmatrix}$

- Define $Var\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix}$ and $Cov\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right) = \begin{bmatrix} Cov(X_1, Y_1) & Cov(X_1, Y_2) \\ Cov(X_2, Y_1) & Cov(X_2, Y_2) \end{bmatrix}$, etc

Outline

1. Random Variables and Probability Distributions✓
2. Means and Variances ✓
3. Identification in Experiments
4. Random Sampling and Sample Means
5. Hypothesis Testing and Inference

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- Suppose that for each person we assign D_i by flipping a coin. This implies that $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0))$. Why?

Using Randomization

- By virtue of the experiment, $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0))$.
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- Similarly, $E[Y_i|D_i = 0] = E[Y_i(0)|D_i = 0] = E[Y_i(0)]$.
- Combining these results, we can see that

$$\underbrace{E[Y_i|D_i = 1]}_{\text{Pop mean for treated}} - \underbrace{E[Y_i|D_i = 0]}_{\text{Pop mean for control}} = \underbrace{E[Y_i(1) - Y_i(0)]}_{\text{Avg treatment effect}} = \tau$$

Thus, the difference in treated/control population means in an experiment identifies the ATE!

Identification under Conditional Unconfoundedness

- Now suppose that $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0)) | \mathbf{X}_i$, where $\mathbf{X}_i$ is a vector of observable characteristic
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Example – Hot Days and Test Scores

- Park et al (2021) study the impact of hot days (D_i) during the school year on test scores (Y_i)
  - Note Their D_i is not binary, although we could imagine a binarized treatment, e.g. $D_i = 1[Hotdays > 10]$
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- Does this seem reasonable to you?

Park et al. (2021) Abstract

Human capital generally, and cognitive skills specifically, play a crucial role in determining economic mobility and macroeconomic growth. While elevated temperatures have been shown to impair short-run cognitive performance, much less is known about whether heat exposure affects the rate of skill formation. We combine standardized achievement data for 58 countries and 12,000 US school districts with detailed weather and academic calendar information to show that the rate of learning decreases with an increase in the number of hot school days. These results provide evidence that climatic differences may contribute to differences in educational achievement both across countries and within countries by socioeconomic status and that may have important implications for the magnitude and functional form of climate damages in coupled human–natural systems.

Example – Returns to College Selectivity

- Dale and Krueger (2002) studied a Q similar to our ongoing example: What is the effect on earnings of attending a selective college?
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- Essentially, they argue that once we know what colleges you applied/ were admitted to, where you choose to go is effectively random
- Do you believe this? Why might this assumption go wrong?
    - Students who choose to go to selective college may still differ in family background, motivation, career plans, etc.

Dale and Krueger (2002)

Abstract

Estimates of the effect of college selectivity on earnings may be biased because elite colleges admit students, in part, based on characteristics that are related to future earnings. We matched students who applied to, and were accepted by, similar colleges to try to eliminate this bias. Using the College and Beyond data set and National Longitudinal Survey of the High School Class of 1972, we find that students who attended more selective colleges earned about the same as students of seemingly comparable ability who attended less selective schools. Children from low-income families, however, earned more if they attended selective colleges.

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- $E[Y_i(1) - Y_i(0) | \mathbf{X}_i = \mathbf{x}]$ is often called the *conditional average treatment effect*, written $CATE(\mathbf{x})$.

Using Conditional Unconfoundedness (cont.)

- We showed that under conditional unconfoundedness, $CATE(\mathbf{x}) = E[Y_i(1) - Y_i(0)|\mathbf{X}_i = \mathbf{x}]$ is identified.
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- Requires $0 < Pr(D_i = 1|\mathbf{X}_i = \mathbf{x}]) < 1$: called an **overlap** condition
- Intuitively, we need there to be some treated and some control units for each value of X_i, in order to learn about the <u>overall</u> ATE

Learning about Population Means

- We just showed that, in an experiment, the average treatment effect is identified as the difference in population means:

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- We need to learn about these **estimands** from the observed sample

- Enter **statistical inference...**

Outline

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  - In the Dewey v. Truman example, not representative!

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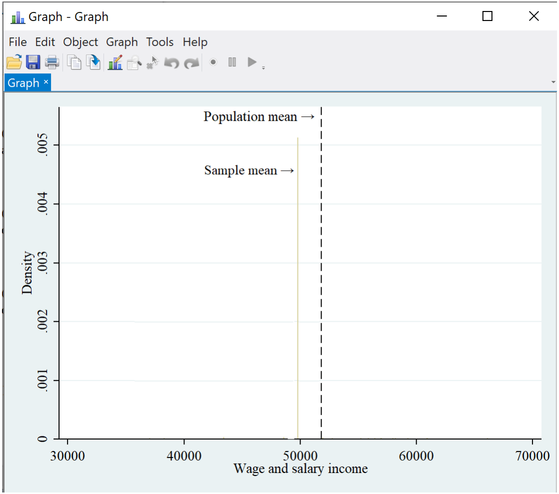
- Equation (1) says that $\hat{\mu}$ is *unbiased*: its average value is μ
- Equation (2) says that the standard deviation of $\hat{\mu}$ from its mean (i.e. μ) shrinks with the sample size N ($\approx$ *consistency*)

Random Sampling and Sample Means

Simulating unbiasedness and consistency for estimating mean income:

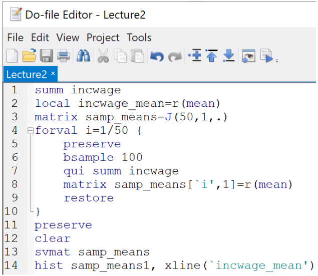
Do-file Editor - Lecture2

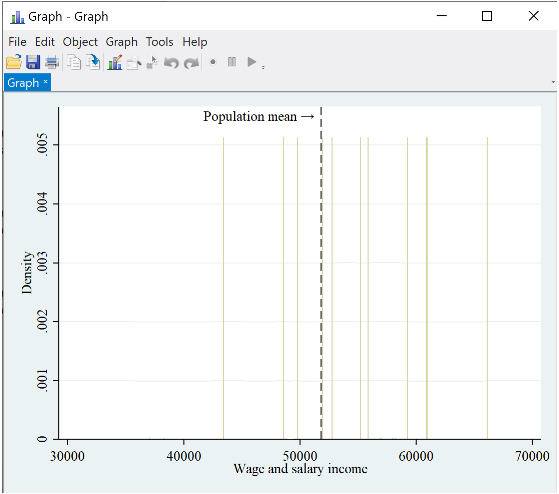
```
summ incwage
local incwage_mean=r(mean)
matrix samp_means=J(50,1,.)
forval i=1/50 {
    preserve
    bsample 100
    qui summ incwage
    matrix samp_means[`i',1]=r(mean)
    restore
}
preserve
clear
svmat samp_means
hist samp_means1, xline(`incwage_mean')
```



Simulations of Random Sampling

Simulating unbiasedness and consistency for estimating mean income:





Random Sampling and Sample Means

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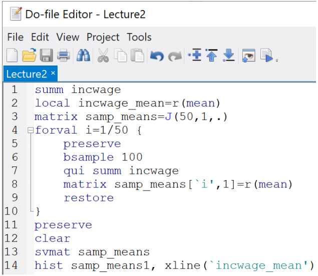

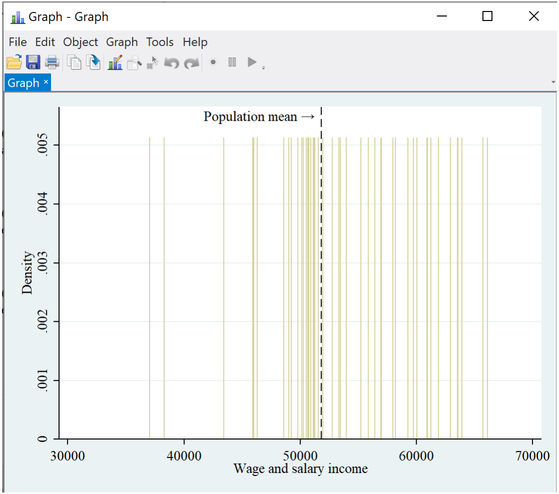

Random Sampling and Sample Means

Simulating unbiasedness and consistency for estimating mean income:

Do-file Editor - Lecture2

```
summ incwage
local incwage_mean=r(mean)
matrix samp_means=J(50,1,.)
forval i=1/50 {
    preserve
    bsample 100
    qui summ incwage
    matrix samp_means[`i',1]=r(mean)
    restore
}
preserve
clear
svmat samp_means
hist samp_means1, xline(`incwage_mean')
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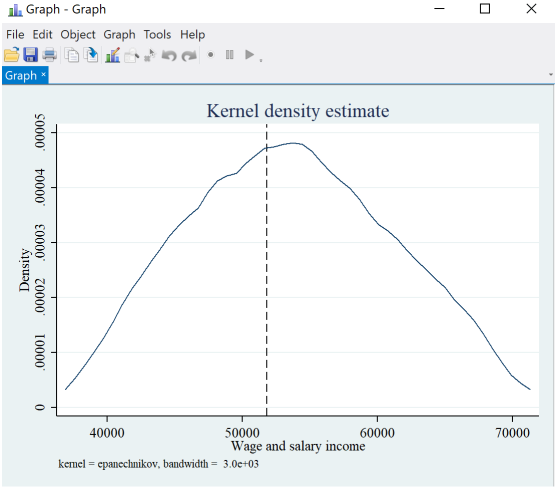

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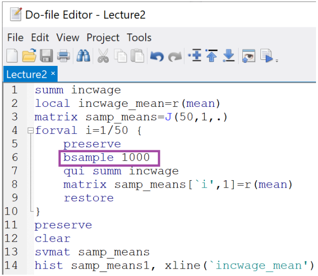


```
summ incwage
local incwage_mean=r(mean)
matrix samp_means=J(50,1,.)
forval i=1/50 {
    preserve
    bsample 1000
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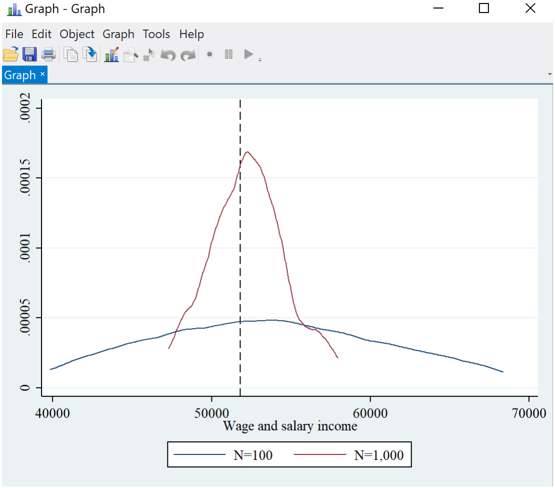

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So: two reasons why $\hat{\mu} = \frac{1}{N}\sum_i Y_i$ is a good estimator of $\mu = E[Y_i]$:

- It is unbiased: $E[\hat{\mu}] = \mu$
- Its variance shrinks to zero as the sample grows: $\lim_{N\to\infty} Var(\hat{\mu}) = 0$

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In the next chapter we'll see another nice property of $\hat{\mu}$: when N is large, its *distribution* is approximately normal

Random Sampling and Sample Means

Given our interest in conditional means $\mu(x) = E[Y_i \mid X_i = x]$, we might also consider conditional sample averages of Y_i given $X_i = x$

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- What if X_i is not discrete, or $N_x \not\to \infty$? Coming soon...

Outline

Hypothesis Testing – an Introduction

- We've shown that when N gets large, the sample mean $\hat{\mu}$ gets close to the population mean μ
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- It tells us whether it is likely to see a sample mean of \$50,000 if the truth is \$55,000, \$70,000, etc.

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1. Specify a **null hypothesis** that the population mean is a particular value, $H_0 : \mu = \mu_0$.
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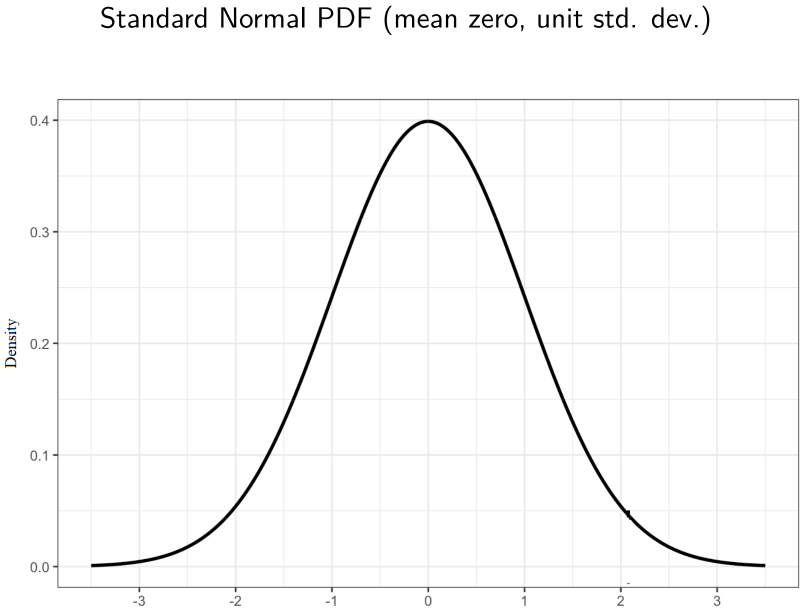
- Intuitively, p is the probability we would see a $|\hat{t}|$ at least this big if the null is true.

Illustration of P-Value Construction



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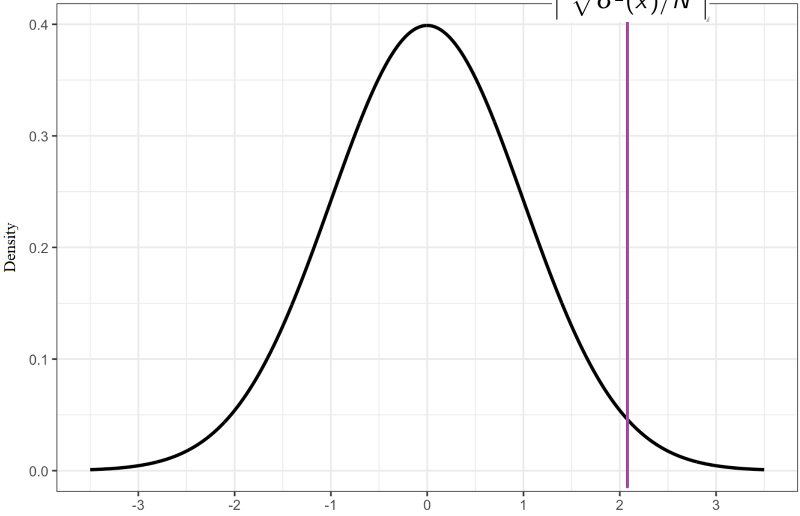
Normalized realization of the random estimator

$$\left| \frac{\hat{\mu}(x)^{obs} - \mu_0}{\sqrt{\sigma^2(x)/N}} \right|$$



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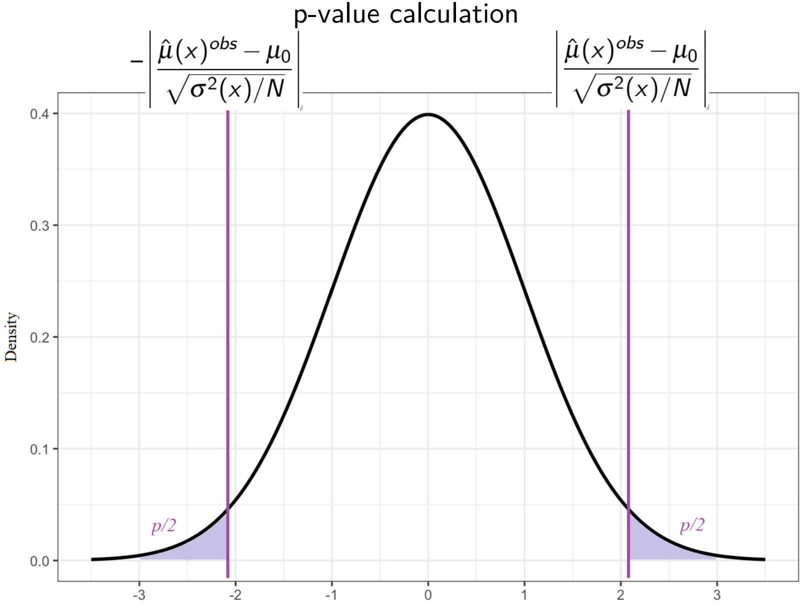

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- The interval $\hat{\mu} \pm 1.96\sigma/\sqrt{N}$ is thus the 95% confidence interval (CI)
  - It has the property that $Pr(\mu_0 \in CI) = 0.95$ when $H_0 : \mu = \mu_0$ is true

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- The *power* of a test is the probability of correctly rejecting the null when it is false (1 - type-II error rate)
  - The power is a function of the *alternative* hypothesis. I.e., the probability that we reject $H_0 : \mu = \mu_0$ when in fact $\mu = \mu_A$

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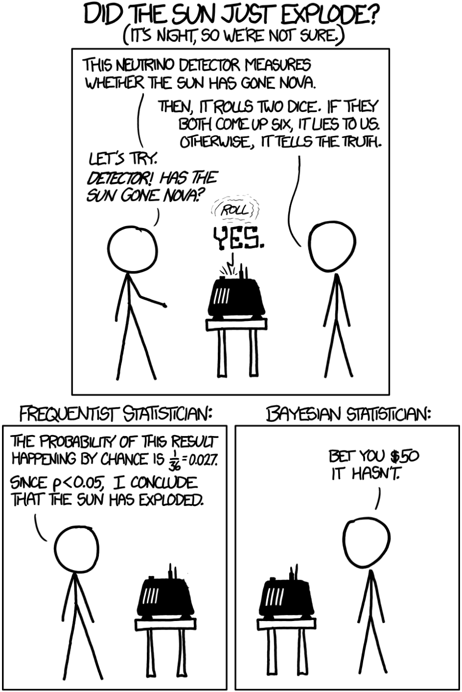
- People often interpret a $p < 0.05$ as strong evidence of an effect and $p \geq 0.05$ as evidence of no effect.
  - But $p = 0.05$ is a fairly arbitrary threshold.
  - It's better to view the p-value as a spectrum indicating how likely is the observed data given the null.

Caution about P-Value Interpretation

- Frequentist p-values are often interpreted as "probability that H_0 is true." Is this right? No!
  - p-value tells us the probability of getting the observed data *assuming* the null is true
  - That is, p-value tells us about $P(data|H_0)$, not $P(H_0|data)$.
  - By Bayes' rule, $P(H_0|Data) = P(Data|H_0) * P(H_0)/P(Data)$. But to formalize this, we need to take a stand on our *prior* belief that H_0 is true, $P(H_0)$.
- People often interpret a $p < 0.05$ as strong evidence of an effect and $p \geq 0.05$ as evidence of no effect.
  - But $p = 0.05$ is a fairly arbitrary threshold.
  - It's better to view the p-value as a spectrum indicating how likely is the observed data given the null.
  - Moreover, p-values can be large even if the null is false (low power!)


DID THE SUN JUST EXPLODE?
(IT'S NIGHT, SO WE'RE NOT SURE.)
THIS NEUTRINO DETECTOR MEASURES WHETHER THE SUN HAS GONE NOVA.
THEN, IT ROLLS TWO DICE. IF THEY BOTH COME UP SIX, IT LIES TO US. OTHERWISE, IT TELLS THE TRUTH.
LET'S TRY.
DETECTOR! HAS THE SUN GONE NOVA?
ROLL
YES.
FREQUENTIST STATISTICIAN:
THE PROBABILITY OF THIS RESULT HAPPENING BY CHANCE IS $\frac{1}{36}$ = 0.027.
SINCE $p < 0.05$, I CONCLUDE THAT THE SUN HAS EXPLODED.
BAYESIAN STATISTICIAN:
BET YOU $50 IT HASN'T.

nature

Published: 26 February 2015

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09 March 2015 This story originally asserted that "The closer to zero the P value gets, the greater the chance the null hypothesis is false." P values do not give the probability that a null hypothesis is false, they give the probability of obtaining data at least as extreme as those observed, if the null hypothesis was true. It is by convention that smaller P values are interpreted as stronger evidence that the null hypothesis is false. The text has been changed to reflect this.

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- We will next review powerful **asymptotic** results. These will allow us to apply similar inference tools if the sample is "large" even when Y_i is not normally distributed.