Chapter 4: Introduction to Regression

Jonathan Roth

Mathematical Econometrics I
Brown University

Motivation

- We showed that under conditional unconfoundedness we can learn the conditional average treatment effect (CATE) by comparing outcome means for the treatment/control group conditional on X_i:

$$CATE(x) = E[Y_i|D_i = 1, X_i = x] - E[Y_i|D_i = 0, X_i = x]$$

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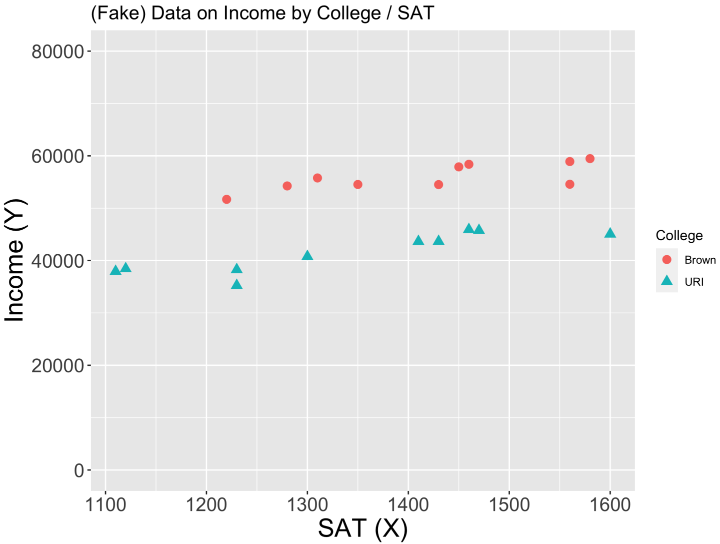
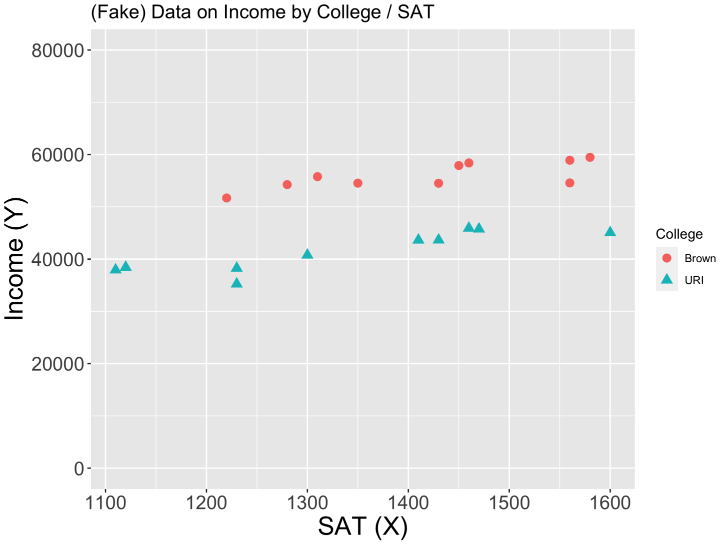
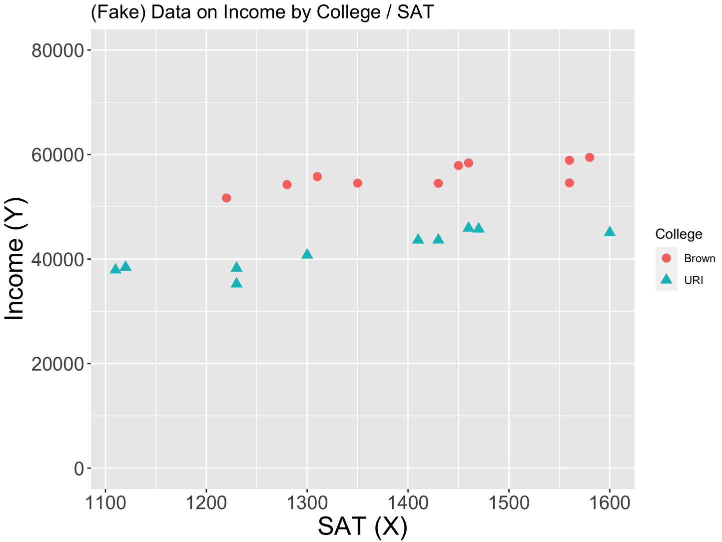
- But what about when X_i is continuous?



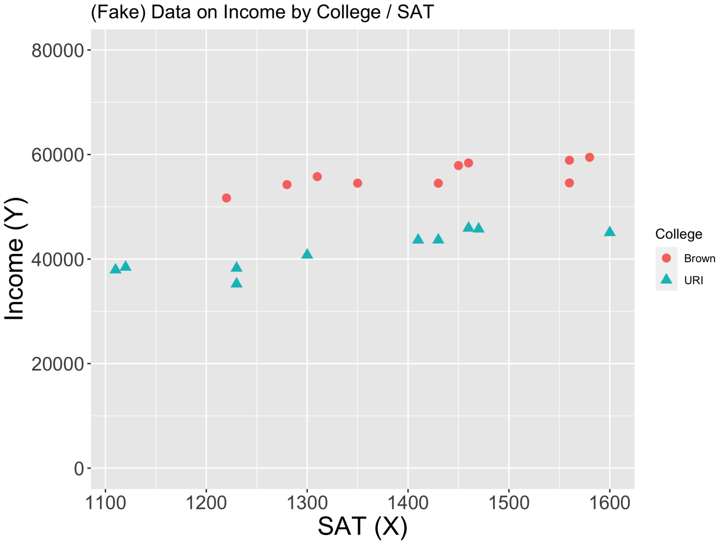
- Suppose this is our data



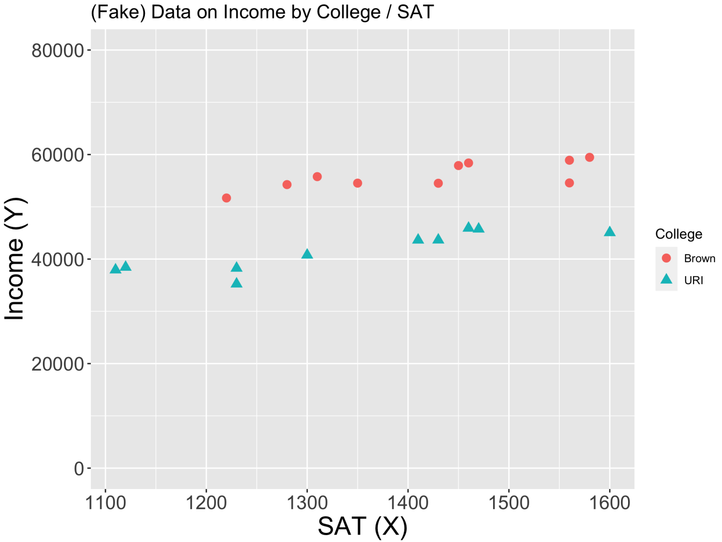
- We might be willing to assume that college attendance is as-good-as-random conditional on SAT score



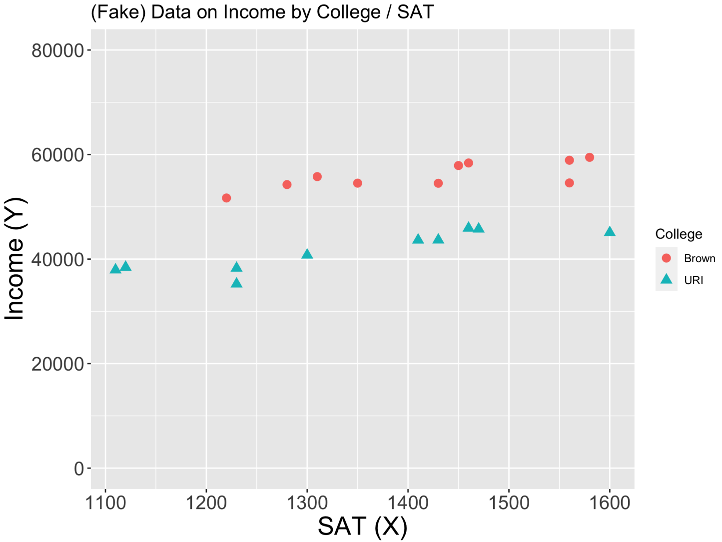
- Say we're interested in the CATE at $X = 1350$. Theory tells us to compare average income for Brown/URI SAT scores with $X = 1350$.



- We could estimate the average at Brown using our 1 student with $X = 1350$. But that estimate is very noisy, & we can't apply the CLT



- We could estimate the average at Brown using our 1 student with $X = 1350$. But that estimate is very noisy, & we can't apply the CLT
- Moreover, we don't have any URI students with $X = 1350$!



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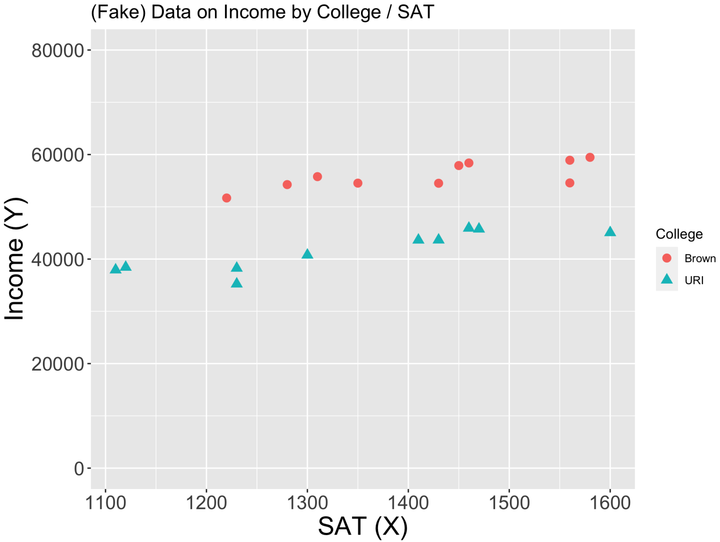

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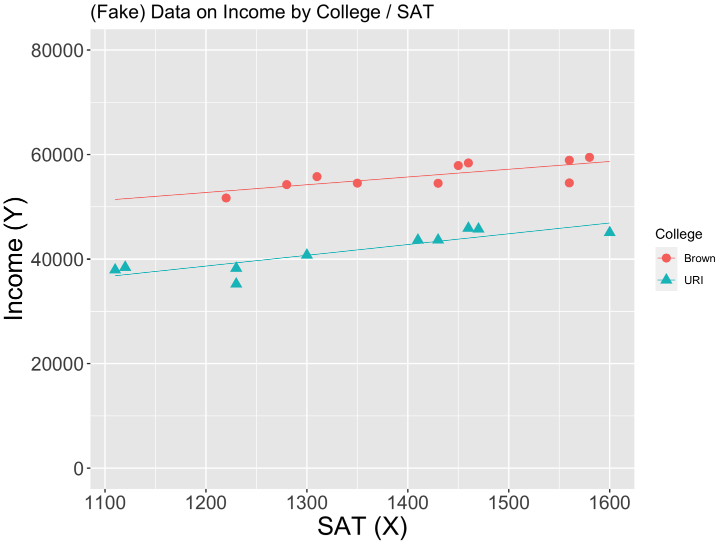
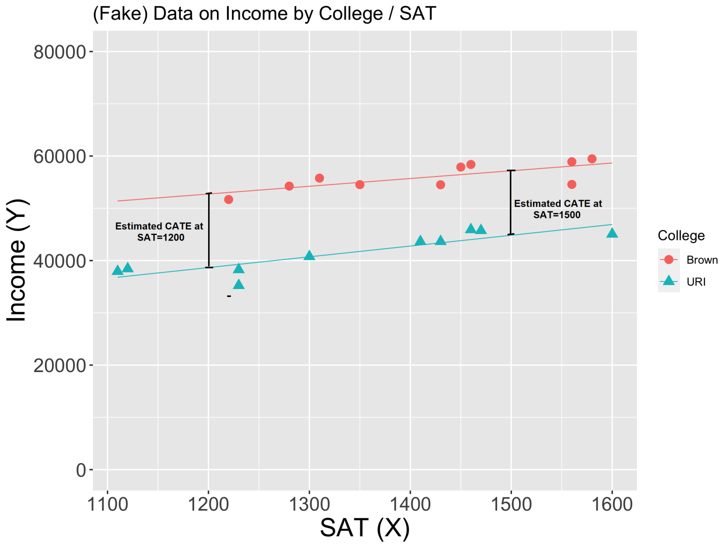


- Clearly, we need to extrapolate from students with other SAT scores.
- What would you do if you were eyeballing it?
- Probably draw a line through the points to estimate the CEFs!



- With these CEF estimates in hand, we can estimate $CATE(x)$ at any x

Outline

1. Population Regression
2. Sample Regressions (OLS)
3. Putting Regression into Practice

Introduction to Regression

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- What happens if the real CEF doesn't take the form we've used for estimation (e.g. isn't linear)?
- We'll try to answer all of these questions over the next several lectures!

Roadmap

- **What we know how to do:** Estimate and test hypotheses about population means using sample means
- **What we want to do:** Estimate approximations to the CEF and test hypotheses about them

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- 4) Argue that even if our assumption about the form of the CEF is wrong, the parameters α, β may provide a "good" approximation.

The "Least Squares" Problem

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- Where does this come from?!

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  Setting the derivative to 0, we obtain

$$E[2(Y_i - \mu)] = 0 \Rightarrow 2E[Y_i] = 2u \Rightarrow u = E[Y_i].$$

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  However, our argument on the previous slide implies that the solution is $u(x) = E[Y_i|X_i = x]$.

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- The minimization above was over *all* functions $u(\cdot)$, including linear ones of the form $a + bx$. Hence,

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- This implies that (α, β) solve

$$\min_{a,b} E[(Y_i - (a + bX_i))^2],$$

as we wanted to show

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- We now have 2 equations with 2 unknowns, which we can use to solve for the CEF parameters (α, β)

The Least Squares Solution

- The solution to the system of equations is as follows:

$$\beta = \frac{E[(X_i - E[X_i])(Y_i - E[Y_i])]}{E[(X_i - E[X_i])^2]} = \frac{Cov(X_i, Y_i)}{Var(X_i)}$$

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- We can therefore use the tools from previous lectures to estimate them and test hypotheses about the CEF!

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$$E[Y_i|X_i = x] = \alpha + x\beta \qquad \checkmark$$

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2. Sample Regressions (OLS)
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Estimating Regression Coefficients

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$$\hat{\beta} = \frac{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2} = \frac{\widehat{Cov}(X_i, Y_i)}{\widehat{Var}(X_i)}$$

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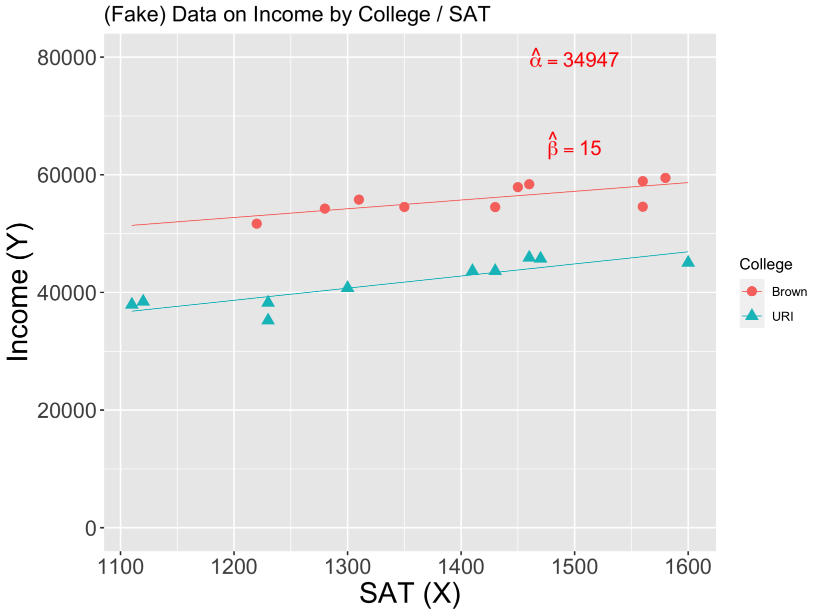
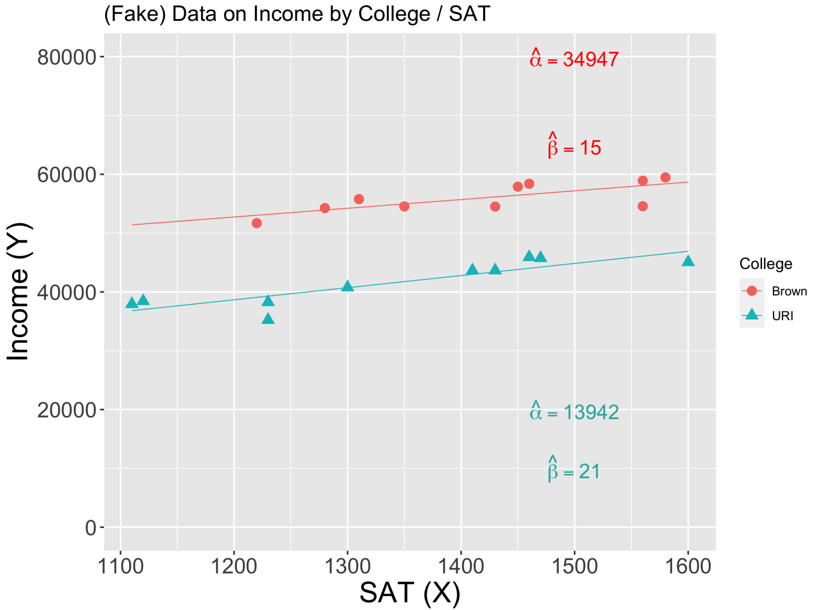
$$\hat{\alpha} = \bar{Y} - \bar{X}\hat{\beta}$$

- These $\hat{\alpha}, \hat{\beta}$ are called *ordinary least squares* (OLS) coefficients
  - They solve the "sample analog" problem, $\min_{a,b} \frac{1}{N}\sum_i (Y_i - (a + bX_i))^2$



(Fake) Data on Income by College / SAT
$\hat{\alpha}$ = 34947
$\hat{\beta}$ = 15
Income (Y)
SAT (X)
80000
60000
40000
20000
0
1100
1200
1300
1400
1500
1600
College
Brown
URI



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$\hat{\beta} = 21$
Income (Y)
SAT (X)
80000
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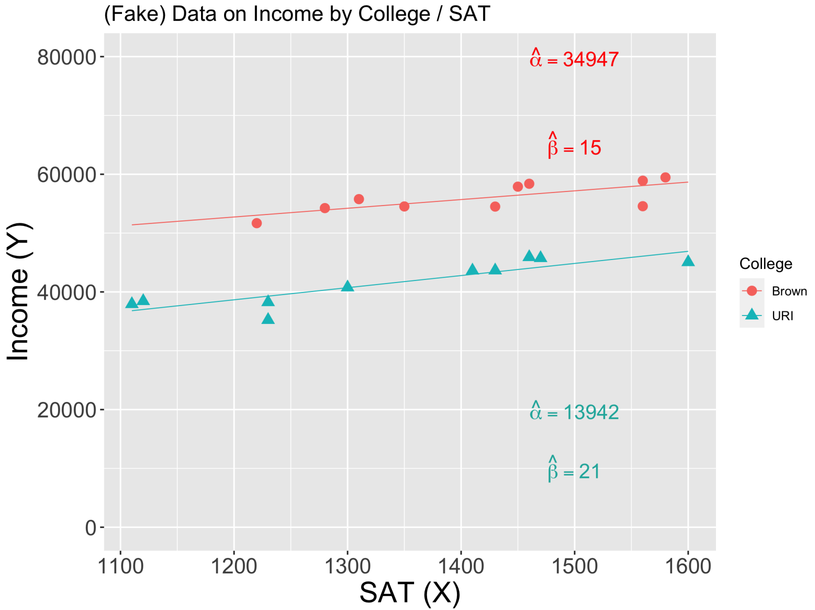
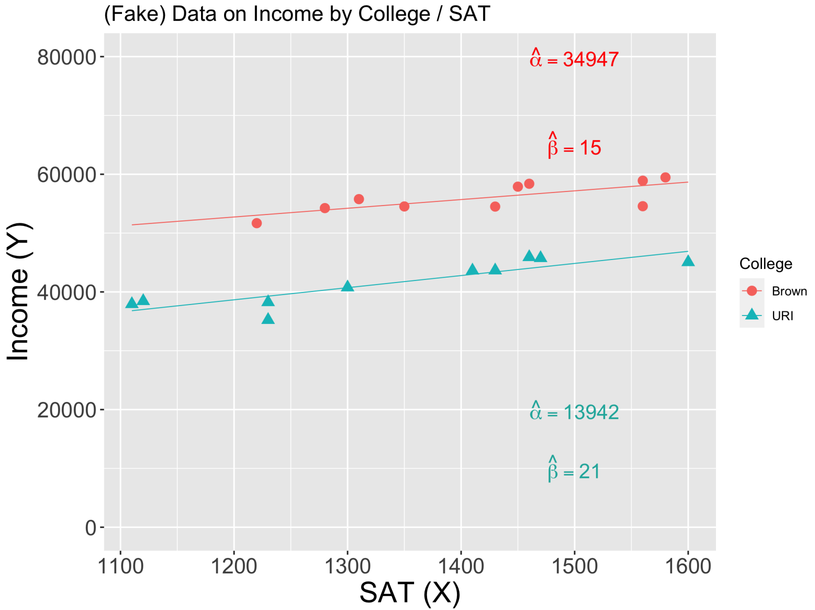


- What is the estimated value of $E[Y_i | D_i = 1, X_i = 1350]$?



- What is the estimated value of $E[Y_i|D_i = 1, X_i = 1350]$?
  $\hat{\alpha} + \hat{\beta} \cdot 1350 = 34947 + 15 \cdot 1350 = 55197$.

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- Analogously, we can show that $\hat{\alpha} \to_p \alpha$.

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- Our OLS estimates $\hat{\alpha}, \hat{\beta}$ are continuous functions of sample means.

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- This is useful because we can then form CIs for β of the form $\hat{\beta} \pm 1.96\hat{\sigma}/\sqrt{N}$, where $\hat{\sigma}$ is our estimate of σ.

Deriving the Asymptotic Distribution for OLS

- Define the **regression residual** $\varepsilon_i = Y_i - (\alpha + X_i\beta)$, implying

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Asymptotic Distribution for OLS (cont.)

- We just derived that $Y_i - \bar{Y} = (X_i - \bar{X})\beta + (\varepsilon_i - \bar{\varepsilon})$.
- Thus,

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\end{aligned}$$

Asymptotic Distribution for OLS (cont.)

- Hence,

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- By LLN and CMT, $\left(\frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})^2\right)^{-1} \to_p Var(X_i)^{-1}$

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- By LLN, CLT, and Slutsky, $\bar{\varepsilon}\sqrt{N}(\bar{X}-E[X_i]) \to_d 0\times\mathrm{N}(0, Var(X_i)) = 0$

Finishing the Asymptotics (!)

- Putting all the pieces together, we see that

$$\sqrt{N}(\hat{\beta} - \beta) \rightarrow_d N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{Var((X_i - E[X_i])\varepsilon_i)}{Var(X_i)^2}$$

- As before, we can estimate the variance σ^2 using sample averages,

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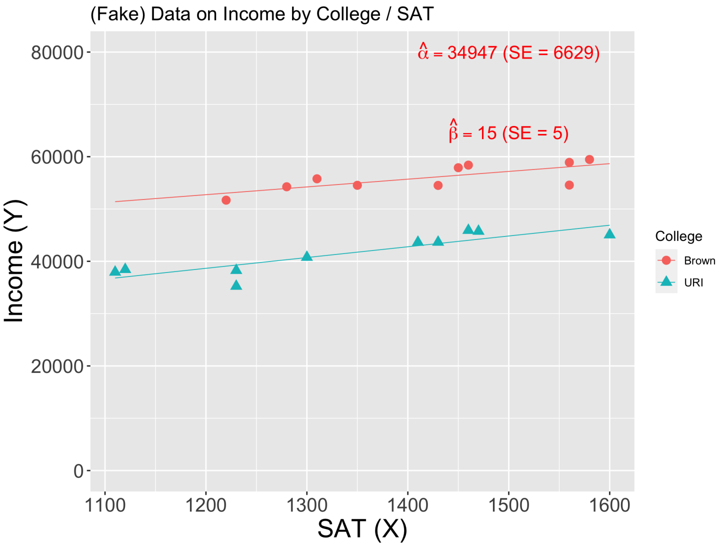
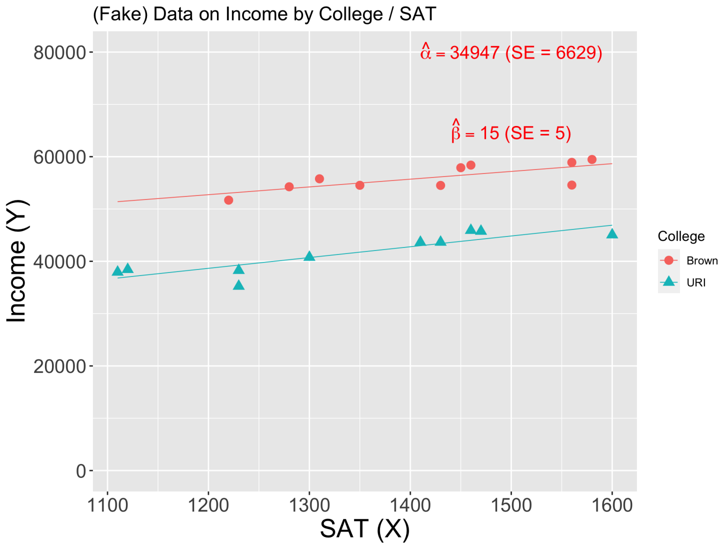
- Can do similar steps to show $\hat{\alpha}$ is asymptotically normally distributed as well. (We'll show formulas later!)



(Fake) Data on Income by College / SAT
α̂ = 34947 (SE = 6629)
β̂ = 15 (SE = 5)
80000
60000
40000
20000
0
Income (Y)
1100
1200
1300
1400
1500
1600
SAT (X)
College
Brown
URI



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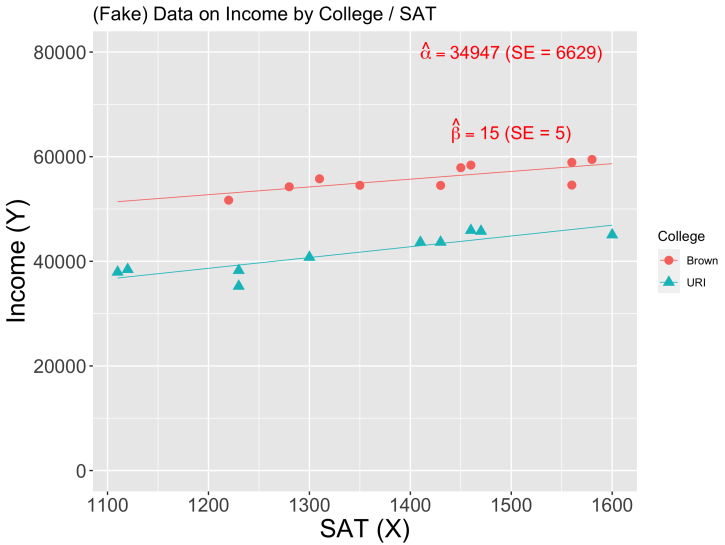

- A CI for β is $\hat{\beta} \pm 1.96 \times SE \approx [5, 25]$

Aside on notation/terminology

- Oftentimes people will say: consider the (population) regression

$$Y_i = \alpha + \beta D_i + \varepsilon_i \tag{1}$$

- What they mean is: "define $(\alpha, \beta) = \arg\min_{a,b} E[(Y_i - (a + bX_i))^2]$"

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- (α, β) are referred to as the "population regression coefficients"
- Likewise, people will say "We estimate equation (1) by OLS" to mean that they compute the sample analogs to α, β via OLS, i.e. $\hat{\alpha}, \hat{\beta}$.

Outline

1. Population Regression✓
2. Sample Regressions (OLS)✓
3. Putting Regression into Practice

Using Regressions to Analyze RCTs

- Recall that when we have an experiment, the average treatment effect is identified by a different in means:

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- Observe that we can write:

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- Analogously, the OLS slope coefficient $\hat{\beta}$ is the difference in sample means which estimates the ATE: $\hat{\beta} = \bar{Y}_1 - \bar{Y}_0 = \hat{\tau}$.

- We can thus use OLS as a convenient tool for estimating the ATE and getting standard errors

Example - WorkAdvance

- Background: gaps between college-educated and non-college educated workers have widened over time

- Yet not everyone thrives in a traditional college background

- **WorkAdvance** is a job-training program intended to provide people with certifiable skills in high-wage industries (e.g. IT, healthcare manufacturing)

- MDRC conducted a randomized trial that randomized access to the training program among people who passed the initial screening

TABLE 1

WORKADVANCE PROVIDERS AND SAMPLE COMPOSITION AT BASELINE

	PER SCHOLAS	ST. NICKS ALLIANCE	MADISON STRATEGIES GROUP	TOWARDS EMPLOYMENT
Provider characteristics				
Location	Bronx, NY	Brooklyn, NY	Tulsa, OK	Northeast Ohio
Target sector(s)	Information technology	Environmental remediation	Transportation, manufacturing	Health care, manufacturing
Approach	Training first	Training first	Training and placement first until fall 2012; then mostly training first	Training and placement first until fall 2012; then mostly training first
Sample composition				
Average age	31	35	35	35
Female (%)	13	15	16	59
Some college or more (%)	63	44	58	57
Currently/ever employed (%)	13/96	11/98	27/99	27/97

- Estimate OLS regression:

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  $\hat{\beta} \pm 1.96 \times SE_{\beta} = 1965 \pm 1.96 \times 609 = [771, 3159]$

- What is the estimated control mean?

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  $\hat{\beta} \pm 1.96 \times SE_{\beta} = 1965 \pm 1.96 \times 609 = [771, 3159]$

- What is the estimated control mean? $\hat{\alpha} = 14636$

Roadmap

- **What we know how to do:** Estimate and test hypotheses about population means using sample means
- **What we want to do:** Estimate approximations to the CEF and test hypotheses about them

How can we use what know to do what we want?

- 1) Assume the CEF takes a particular form, e.g. linear:

$$E[Y_i|X_i = x] = \alpha + x\beta \qquad \checkmark$$

- 2) Show that under this assumption, α and β can be represented as functions of population means. ✓
- 3) Use our tools for estimating population means using sample means to estimate α, β and test hypotheses about them. ✓
- 4) Argue that even if our assumption about the form of the CEF is wrong, the parameters α, β may provide a “good” approximation.

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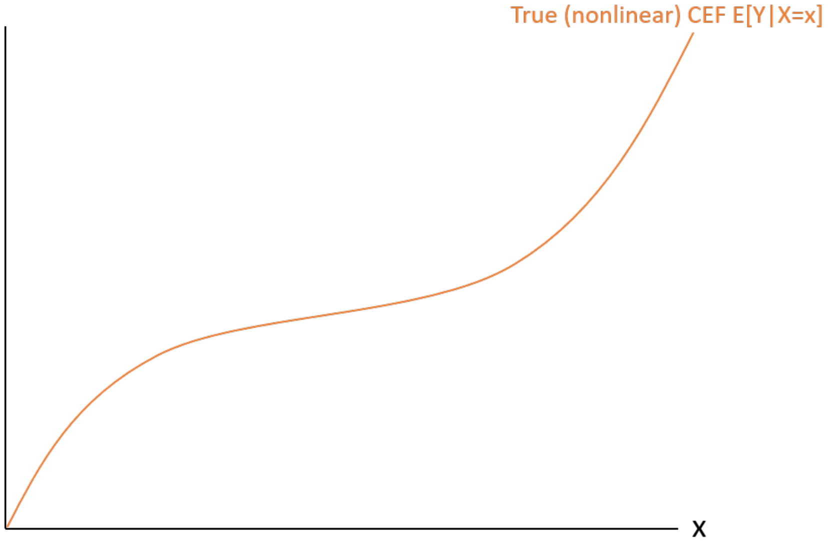
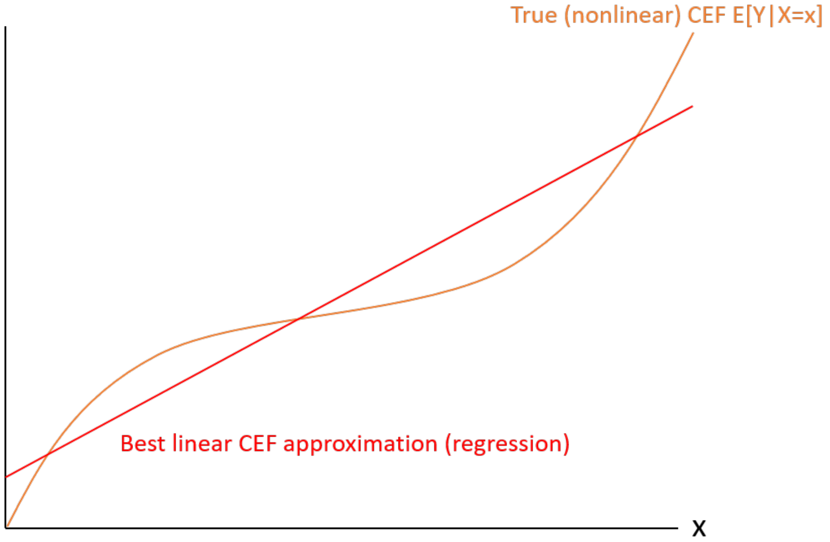
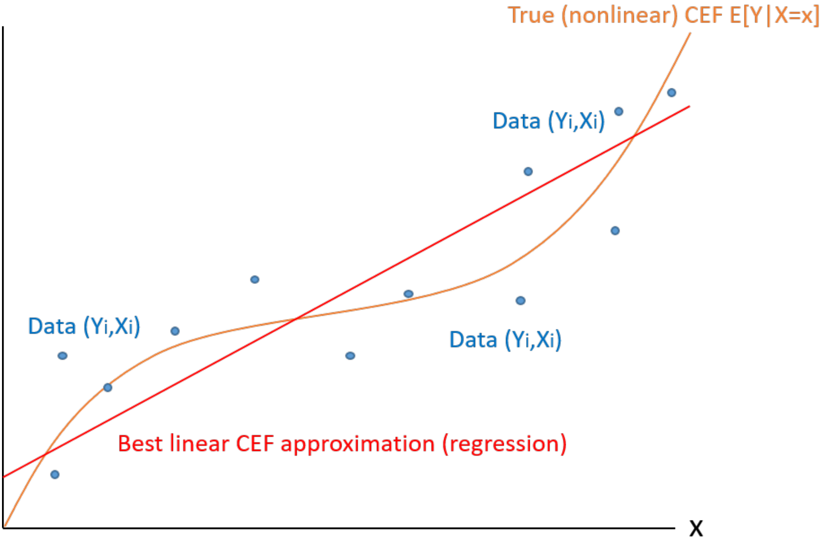
  That is, we get the linear function that's "closest" to the CEF in terms of mean-squared error


True (nonlinear) CEF E[Y|X=x]
X


True (nonlinear) CEF E[Y|X=x]
Best linear CEF approximation (regression)
X


True (nonlinear) CEF E[Y|X=x]
Data (Yi,Xi)
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Best linear CEF approximation (regression)
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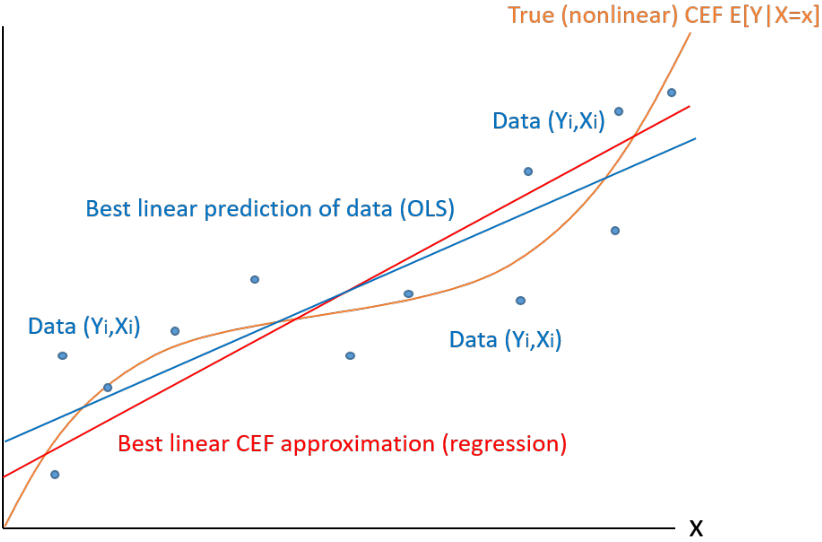
True (nonlinear) CEF E[Y|X=x]
Data (Yi,Xi)
Best linear prediction of data (OLS)
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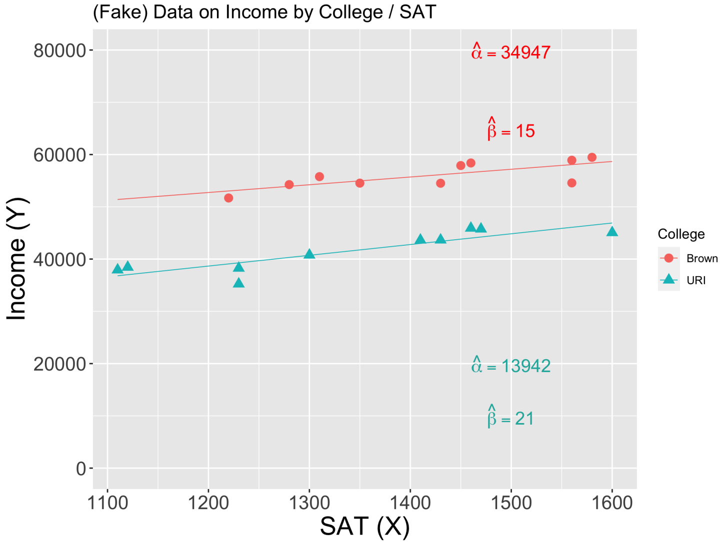
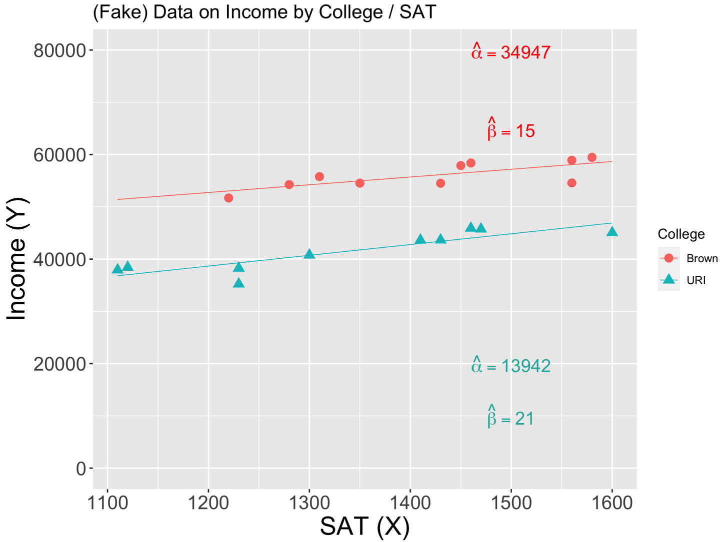
- Hence, $E[(Y-(\alpha+\beta X))^2] = [(Y-\mu(X))^2]+E[\mu(X)-(\alpha+\beta X))^2]$. But the first term doesn't depend on β. So minimizing $E[(Y-(\alpha+\beta X))^2]$ is the same as minimizing $E[\mu(X)-(\alpha+\beta X))^2]$



- $E[Y_i|D_i=1,X_i=x]-E[Y_i|D_i=0,X_i=x]\approx\alpha_1+\beta_1x-(\alpha_0+\beta_0x);$



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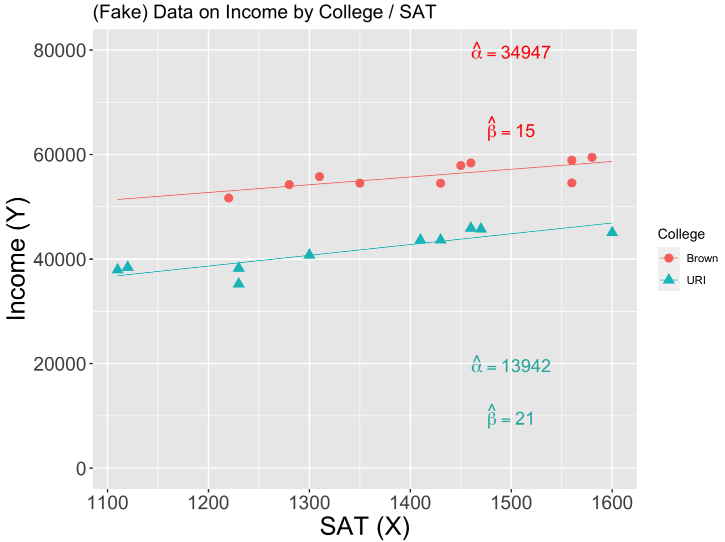

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- So w/conditional ignorability, $ATE = E[CATE(X_i)] \approx 21,005 - 6E[X_i]$